Axisymmetric Magnetohydrodynamic Equilibria without a Wall

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Starting from the ideal magnetohydrodynamic (MHD) equations, we consider the following axisymmetric configuration: a current-carrying plasma torus in a homogeneous magnetic field that is aligned parallel to the torus axis. At a certain field strength this configuration is in equilibrium without need of external current singularities such as wires or walls.The magnetic flux function is expanded in small inverse aspect ratio. The geometry of this configuration is completely determined to second order as a function of the profile parameters.

1. Introduction

1.1. Derivation of the Equation

The magnetic field of any stationary current distribution can be completely described in SI units by the two equations

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}, \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 . \tag{2}$$

The solenoidal property (2) is satisfied by the ansatz

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} . \tag{3}$$

The vector potential $\boldsymbol{A}$ here is determined apart from the gradient of a scalar function. With the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ it holds that

$$\Delta \boldsymbol{A}(\boldsymbol{r}) = -\mu_0 \boldsymbol{j}(\boldsymbol{r}) . \tag{4}$$

The solution of this second-order differential equation is Biot-Savart's formula familiar from electrodynamics:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \, \mathrm{d}V' . \tag{5}$$

Axisymmetric magnetic fields can be completely split into a poloidal and a toroidal component. The poloidal field can be expressed by the toroidal component A_φ of the vector potential and, if $\nabla \times \boldsymbol{A}$ is to describe just a poloidal field component, $\boldsymbol{A}$ itself should have just a toroidal component:

$$\boldsymbol{A}_\varphi = -\psi(R, z) \nabla\varphi . \tag{6}$$

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Like all scalar functions in an axisymmetric configuration, the proportionality factor $\psi(R, z)$ is independent of the toroidal angle φ. Furthermore, for the curl it follows that

$$\boldsymbol{B}_{\text{pol}} = \nabla\varphi \times \nabla\psi .$$

Solving (6) for ψ yields

$$\psi(\boldsymbol{r}) = -R A_\varphi(\boldsymbol{r}) . \tag{7}$$

This means that direct calculation of the flux function $\psi(\boldsymbol{r})$ requires that only the φ-component of the vector potential $\boldsymbol{A}(\boldsymbol{r})$ of the current distribution be known. According to Biot-Savart it is generated solely by the toroidal current component:

$$A_\varphi(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_V \frac{j_\varphi(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \, \mathrm{d}V' . \tag{8}$$

This is converted to the integral over the plasma cross-section D [1]:

$$A_\varphi = \frac{\mu_0}{\pi} \iint_D \sqrt{R'/R} \, \frac{1}{k} \left[\left(1 - \tfrac{1}{2} k^2\right) K(k) - E(k)\right] \cdot j_\varphi(R', z') \, \mathrm{d}R' \, \mathrm{d}z' ,$$

with the complete elliptic integrals of the first and second kinds

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta} \, \mathrm{d}\theta$$

and

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

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as well as the so-called module of the integrals

$$k^2 = \frac{4RR'}{(R+R')^2 + (z-z')^2}.$$

Together with (7) this yields an expression for the flux function ψ of a toroidal configuration with the φ-component of the equilibrium current density $j_\varphi(R', z')$ and Green's function $G(R, R', z, z')$ [2]:

$$\psi(R, z) = -\mu_0 \iint_D G(R, R', z, z')\, j_\varphi(R', z')\, \mathrm{d}R'\, \mathrm{d}z' \quad (9)$$

with

$$G(R, R', z, z') = \frac{1}{\pi k} \sqrt{R'R}\, [(1 - \tfrac{1}{2}k^2)\, K(k) - E(k)]$$

and

$$j_\varphi(R, z) = -\frac{1}{R}\left(\mu_0 I \frac{\mathrm{d}I}{\mathrm{d}\psi} + R^2 \frac{\mathrm{d}p}{\mathrm{d}\psi}\right).$$

If the plasma torus is left to itself, it will expand as a result of the radial forces exerted by the pressure gradient and the product of the plasma current and magnetic field. These so-called hoop forces can be compensated by an external homogeneous magnetic field parallel to the torus axis which interacts with the toroidal current and exerts a counteracting force.

At infinity the magnetic field generated by the currents flowing in the torus vanishes, so that only the homogeneous field is left, for which the notation $\boldsymbol{B}_z$ is used.

In dimensional quantities, it holds that

$$\psi(R) = -\tfrac{1}{2} B_z R^2 + c, \quad (10)$$

Let $R_{\max}$ and $R_{\min}$ be the largest and lowest value of R, respectively, for the plasma edge and

$$\bar{r} = \tfrac{1}{2}(R_{\max} - R_{\min}), \quad \bar{R} = \tfrac{1}{2}(R_{\max} + R_{\min}).$$

We call $\varepsilon = \bar{r}/\bar{R}$ the inverse aspect ratio. Let

$$J = \iint_D j_\varphi(R, z)\, \mathrm{d}R\, \mathrm{d}z \quad (11)$$

be the total toroidal current and a, b, x, y, and symbols with a hat denote dimensionless quantities. We then put

$$R = \bar{R}(1+\varepsilon x), \quad z = \bar{r} y, \quad \psi = -\mu_0 \bar{R} J \hat{\psi},$$

$$j_\varphi = -\frac{J}{\bar{r}^2} \hat{j}_\varphi, \quad G = \bar{R}\hat{G}, \quad c = -\bar{R} J \mu_0 a,$$

$$B_z = 2\frac{J}{\bar{R}} \mu_0 b.$$

Note that $J < 0$ if $\boldsymbol{B}_{\mathrm{pol}}$ is going around the magnetic axis in the mathematically positive sense.

The complete flux function consists of the component generated by the current flowing in the plasma and the component generated by the external field. After dropping the hats the dimensionless integral equation reads

$$\psi(x, y) = \psi_{\mathrm{Pl}}(x, y) + \psi_{\mathrm{ext}}(x, y) \quad (12\text{a})$$

$$= -\iint_D G(x, x', y, y')\, j_\varphi(x', y')\, \mathrm{d}x'\, \mathrm{d}y' + b(1+\varepsilon x)^2 + a.$$

The solution of (12a) has to satisfy two conditions. We describe the plasma edge with the relation

$$\psi|_{\Gamma_{\mathrm{PV}}} = 0.$$

In the interior of the plasma the flux function is assumed to be positive. Once and for all, we fix the intersection of the plasma edge with the R-axis at the coordinates $x = \pm 1$. This requirement is called the subsidiary condition (B):

$$\text{(B)} \quad \psi(x=1, y=0) = \psi(x=-1, y=0) = 0. \quad (12\text{b})$$

As second condition we normalize the total toroidal current to 1, as usual:

$$\text{(N)} \quad 1 = \iint_D j_\varphi(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{\sigma} = \int_{\partial D} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = -\int_{\partial D} \frac{\partial \psi}{\mathrm{d}n} \frac{\mathrm{d}l}{R}. \quad (12\text{c})$$

1.2. Linear Profile Functions

The profile functions p' and $I'I$ can be given arbitrarily to a certain extent. They are chosen linear in ψ:

$$\frac{\mathrm{d}p}{\mathrm{d}\psi} = \beta_{\mathrm{p}}(\alpha^2 \psi + \xi), \quad (13\text{a})$$

$$I\frac{\mathrm{d}I}{\mathrm{d}\psi} = (1 - \beta_{\mathrm{p}})(\alpha^2 \psi + \eta), \quad (13\text{b})$$

with the abbreviations

$$\xi = \lambda + (\beta_{\mathrm{P}} - 1)\, v, \quad (13\text{c})$$

$$\eta = \lambda + \beta_{\mathrm{P}}\, v, \quad (13\text{d})$$

$$\text{and} \quad \lambda = (\xi - \eta)\, \beta_{\mathrm{P}} + \eta \quad (13\text{e})$$

and with the poloidal beta

$$\beta_{\mathrm{P}} = \frac{\langle p \rangle}{\langle B_{\mathrm{pol}}^2 \rangle / 2}; \quad \langle \ldots \rangle = \int \ldots \mathrm{d}\psi.$$

$\beta_{\mathrm{p}} - 0$ is the force-free case with vanishing pressure gradient and $\boldsymbol{j}$ parallel to $\boldsymbol{B}$. The term $\beta_{\mathrm{P}}\xi$ describes the limit of the pressure gradient when approaching the plasma edge from the magnetic axis. For $\beta_P \neq 0$ and non-vanishing ξ the pressure gradient is thus discontinuous at the plasma edge. Since $p \geq 0$ it holds that $\xi \geq 0$.

I' is proportional to the poloidal current density. The term $(1-\beta_P)\eta$ is proportional to the poloidal current density at the plasma edge. In the special case $\eta = 0$ it vanishes at the edge, and in the case $\beta_P = 1$ it vanishes everywhere; the toroidal magnetic field is then a vacuum field that decreases as $1/R$. The parameter α^2 can be chosen arbitrarily between $-\infty$ and an upper bound yet to be defined.

It will be seen that λ is fixed in zeroth and first order whenever the current profile parameter α is given. We then have v for varying the values of the profile functions at the plasma edge $\beta_P \xi$ and $(1-\beta_P)\eta$; of course we do not have complete freedom in choosing ξ and η because they are related to the poloidal beta and the current density profile through (13e).

We thus look for solutions of the integral equation

$$\psi(x, y) = -\iint_D G(x, y, x', y')\, j_\varphi(x', y')\,\mathrm{d}x'\,\mathrm{d}y' + b(1+\varepsilon x)^2 + a \qquad (14a)$$

with the toroidal current density

$$j_\varphi(x, y) = -(1+\varepsilon x)\,\beta_p\big(\alpha^2\psi(x, y) + \lambda + (\beta_P - 1)\,v\big) - \frac{1}{1+\varepsilon x}(1-\beta_p)\big(\alpha^2\psi(x, y) + \lambda + \beta_P v\big)$$

and Green's function G, including the normalization condition

$$1 = -\int_{\partial D} \frac{\partial\psi}{\partial n}\,\frac{\mathrm{d}l}{1+\varepsilon x}, \qquad (14b)$$

and the subsidiary condition

$$\psi(1, 0) = \psi(-1, 0) = 0\,. \qquad (14c)$$

That makes three conditions altogether. These will be used to determine the three parameters a, b, and λ.

1.3. Expansion

First the factors of the integrand $G(x, y, x', y')$ and $j_\varphi(x', y')$ are expanded in the inverse aspect ratio.

Up to and including fourth order the modulus k^2 yields

$$k^2 = 1 - \tfrac{1}{4}\varrho^2\varepsilon^2 + \tfrac{1}{4}(x + x')\,\varrho^2\varepsilon^3 + \tfrac{1}{16}\varrho^2[-3(x+x')^2 + (y - y')^2]\,\varepsilon^4 + O(\varepsilon^5) \quad (15)$$

with

$$\varrho^2 = (x - x')^2 + (y - y')^2\,.$$

In tables of mathematical formulae one finds expansions of elliptical integrals for k about 1 [3]:

$$E(k) = 1 + \tfrac{1}{2}(\Lambda - \tfrac{1}{2})(1 - k^2) + \tfrac{3}{16}(\Lambda - \tfrac{13}{12})(1 - k^2)^2 + O\big((1 - k^2)^3\big), \qquad (16a)$$

$$K(k) = \Lambda + \frac{\Lambda - 1}{4}(1 - k^2) + \tfrac{9}{64}(\Lambda - \tfrac{7}{6})(1 - k^2)^2 + O\big((1 - k^2)^3\big) \qquad (16b)$$

with the abbreviation $\Lambda = \ln\big(4/\sqrt{1-k^2}\big)$.

In the context of a two-scale method we regard ε and $\ln\varepsilon$ as two independent quantities. We take $\ln\varepsilon$ as a constant and expand in ε:

$$\Lambda = \ln\frac{8}{\varrho\,\varepsilon} + \varepsilon\,\frac{x + x'}{2} + \varepsilon^2[-\tfrac{1}{8}(x+x')^2 + \tfrac{1}{8}(y-y')^2] + O(\varepsilon^3)\,. \qquad (17)$$

We require Λ up to and including second order and therefore have to take the expansion of k^2 to fourth order. Substitution of (15) and (17) into (16) yields

$$E(k) = 1 + \varepsilon^2\frac{\varrho^2}{8}\left[\ln\frac{8}{\varrho\,\varepsilon} - \frac{1}{2}\right] + O(\varepsilon^3)\,,$$

$$K(k) = \ln\frac{8}{\varrho\,\varepsilon} + \varepsilon\,\frac{x + x'}{2} + \varepsilon^2\left[\frac{\varrho^2}{16}\left(\ln\frac{8}{\varrho\,\varepsilon} - 1\right) - \frac{1}{8}(x + x')^2 + \frac{1}{8}(y - y')^2\right] + O(\varepsilon^3)\,.$$

This gives an expansion for Green's function $G(R, z, R', z') = G(x, y, x', y')$:

$$G_0 = \frac{1}{2\pi}\left(\ln\frac{8}{\varrho\,\varepsilon} - 2\right), \qquad (18a)$$

$$G_1 = \frac{1}{4\pi}(x + x')\left(\ln\frac{8}{\varrho\,\varepsilon} - 1\right), \qquad (18b)$$

$$G_2 = \frac{1}{16\pi}\left[(x + x')^2 + (y - y')^2 + \frac{\varrho^2}{2}\left(\ln\frac{8}{\varrho\,\varepsilon} + 1\right) + (y - y')^2\left(\ln\frac{8}{\varrho\,\varepsilon} - 2\right)\right]. \qquad (18c)$$

The Taylor series expansion of the toroidal current density in ε yields the following expressions on expanding the parameter λ:

$$-j_{\varphi 0}(x, y) = \alpha^2\psi_0(x, y) + \lambda_0 \qquad (19a)$$

$$-j_{\varphi 1}(x, y) = \alpha^2 \psi_1(x, y) + \lambda_1 + x\{(2\beta_P - 1)(\alpha^2 \psi_0(x, y) + \lambda_0) + 2(\beta_P - 1)\beta_P v\}, \quad (19\text{b})$$

$$-j_{\varphi 2}(x, y) = \alpha^2 \psi_2(x, y) + \lambda_2 + x(2\beta_P - 1)(\alpha^2 \psi_1(x, y) + \lambda_1) + x^2(1 - \beta_P)(\alpha^2 \psi_0(x, y) + \lambda_0 + \beta_P v). \quad (19\text{c})$$

As $\psi(x, y)$ in zeroth order will exhibit cylindrical symmetry, it is convenient to introduce a polar coordinate system (r, θ) with the relations

$$x = r \cos\theta, \quad y = r \sin\theta, \quad dx\, dy = r\, dr\, d\theta.$$

This yields

$$\varrho^2 = r^2 + r'^2 - 2 r r' \cos(\theta - \theta').$$

For the plasma edge it holds that $\psi(r, \theta) = 0$. It is described as a function $\hat{r}(\theta, \varepsilon)$ of the poloidal angle and aspect ratio. To simplify the notation, the ε-dependence will not always be given explicitly. The subsidiary condition (14c) fixes the edge curve at the intersections with the R-axis for all orders:

$$\hat{r}(\theta = 0) = \hat{r}(\theta = \pi) = 1. \quad (20)$$

Altogether then, we expand the following quantities:

$$\psi(\varepsilon, r, \theta) = \psi_0(r) + \varepsilon \psi_1(r, \theta) + \varepsilon^2 \psi_2(r, \theta) + O(\varepsilon^3),$$

$$G(\varepsilon, r, \theta, r', \theta') = G_0(r, \theta, r', \theta') + \varepsilon G_1(r, \theta, r', \theta') + \varepsilon^2 G_2(r, \theta, r', \theta') + O(\varepsilon^3),$$

$$j_\varphi(\varepsilon, r, \theta) = j_{\varphi 0}(r) + \varepsilon j_{\varphi 1}(r, \theta) + \varepsilon^2 j_{\varphi 2}(r, \theta) + O(e^3),$$

$$\hat{r}(\varepsilon, \theta) = \hat{r}_0 + \varepsilon \hat{r}_1(\theta) + \varepsilon^2 \hat{r}_2(\theta) + O(\varepsilon^3),$$

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + O(\varepsilon^3),$$

$$a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + O(\varepsilon^3),$$

$$b(\varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + O(\varepsilon^3)$$

and regard the parameters α, β_P, and v as being arbitrary. Here it must always be ensured – especially with v – that the orders of the expansion are not perturbed, i.e. that v is small compared with $1/\varepsilon$.

Our mathematical problem thus takes the following form:

1. Solution of the nonlinear integral equation

$$\psi(r, \theta) = -\int_0^{2\pi} \int_0^{\hat{r}(\theta')} G(r, r', \theta, \theta')\, j_\varphi(r', \theta')\, r'\, dr'\, d\theta' + b(1 + \varepsilon r \cos\theta)^2 + a, \quad (21\text{a})$$

2. Satisfaction of the normalization condition

$$1 = -\int_{\partial D} \frac{\partial \psi}{\partial n} \frac{dl}{1 + \varepsilon r \cos\theta}, \quad (21\text{b})$$

3. Satisfaction of the subsidiary condition

$$\psi(1, 0) = \psi(1, \pi) = 0. \quad (21\text{c})$$

Since the edge function $\hat{r}(\varepsilon, \theta)$ is one of the unknown quantities to be determined, this corresponds to the solution of a free-boundary problem. The strength of the applied vertical field governs, among other quantities, the aspect ratio. Since, however, we want an expansion in ε, we have to be able to specify and vary the aspect ratio, and not the external field, for example. The vertical field will therefore be matched to ε and not vice versa. Which quantity is specified and which one is matched to it in order to satisfy the equilibrium condition is of no importance; the two approaches are equivalent.

2. Solution

2.1. Zeroth Order

2.1.1. Solution of the Problem

In the case of infinite aspect ratio ($\varepsilon = 0$) we have to solve the equation

$$\psi_0(r) = \int_0^{2\pi} \int_0^1 G_0(r, \theta, r', \theta')\, [\alpha^2 \psi_0(r') + \lambda_0]\, r'\, dr'\, d\theta' + b_0 + a_0 \quad (22\text{a})$$

with due allowance for the normalization condition (see annex B)

$$(\text{N}) \quad \psi_{0,r}(1) = -1/2\pi \quad (22\text{b})$$

and the subsidiary condition

$$(\text{B}) \quad \psi_0(1) = 0. \quad (22\text{c})$$

First we rewrite (22a):

$$\psi_0(r) = \alpha^2 \int_0^{2\pi} \int_0^1 G_0 \psi_0(r')\, r'\, dr'\, d\theta' + \lambda_0 \int_0^{2\pi} \int_0^1 G_0\, r'\, dr'\, d\theta' + b_0 + a_0.$$

We then use the fact that $\psi_0(r)$ is angularly independent and can therefore be eliminated from the θ'-integral. The θ'-integration is now carried out [4], a distinction being made between the cases $r \leq r'$ and $r \geq r'$:

$$\int_0^{2\pi} G_0(r, \theta, r', \theta')\, d\theta' = \ln \frac{8}{\varepsilon} - 2 + K_0(r, r')$$

with

$$K_0(r, r') = \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{1}{\varrho}\, d\theta' = \begin{cases} -\ln r', & r \leq r', \\ -\ln r, & r \geq r'. \end{cases}$$

This then gives us the one-dimensional integral equation

$$\psi_0(r) = \alpha^2 \int_0^1 K_0(r, r')\, \psi_0(r')\, r'\, dr' + \alpha^2 \left(\ln \frac{8}{\varepsilon} - 2\right) \int_0^1 \psi_0(r')\, r'\, dr' + \lambda_0 \left[\frac{1}{2}\left(\ln \frac{8}{\varepsilon} - 2\right) + \int_0^1 K_0(r, r')\, r'\, dr'\right] + b_0 + a_0 . \tag{23}$$

Depending on the sign of α^2 various solutions are now obtained. We distinguish three cases:

$\alpha^2 > 0$, α real

$\alpha^2 < 0$, α imaginary,

$\alpha^2 = 0$.

In addition, α^2 is assumed to have an upper bound. The existence of such a bound is deduced by the following reasoning:

If the flux function has extrema in the interior of the plasma which are not *O*-points, any perturbation occurring can lead to the formation of so-called magnetic islands, which alter the topology of the flux function. The flux surfaces are then no longer simply nested. Various islands can interact and destroy the magnetic surfaces. The magnetic field lines then occupy regions in which there is enhanced radial transport of charged particles, which very severely impairs the magnetic confinement [5].

In order to rule out such effects from the outset, it is sufficient to assume the flux function in zeroth order to be monotonic. What influence this requirement has on the current profile parameter α will be seen as soon as $\psi_0(r)$ is known.

We now consider the following theorem on differential and integral equations:

Let the differential equation

$$L[u(x)] + \lambda \varrho u(x) = \chi(x)$$

be given, where $L[u]$ is a linear differential operator of second order, λ a parameter, $\chi(x)$ a piecewise continuous and $\varrho(x)$ a positive continuous function. Let u satisfy certain boundary conditions. If $K(x, \xi)$ is the Green's function assumed to exist for $L[u]$ which satisfies the boundary conditions, then the integral equation

$$u(x) = \lambda \int K(x, \xi)\, \varrho(\xi)\, u(\xi)\, d\xi + g(x)$$

with

$$g(x) = -\int K(x, \xi)\, \chi(\xi)\, d\xi$$

is equivalent to the above differential equation. In other words, every solution of the differential equation is also a solution of the integral equation and vice versa. In particular, the function

$$u(x) = \int K(x, \xi)\, \varphi(\xi)\, d\xi$$

satisfies the differential equation

$$L[u] = -\varphi(x)$$

with the boundary conditions. Furthermore, if $L[u]$ is self-adjoint, the kernel of the integral equation $K(x, \xi)$ is symmetric with respect to interchange of parameter and argument: $K(x, \xi) = K(\xi, x)$ [6].

Obviously, we can transform an integral equation of type (23) into an equivalent differential equation plus the appropriate boundary condition by using a suitable differential operator L.

We now use the fact that the Green's function of the differential expression belonging to the zeroth order Bessel function

$$L_0[u] = r u_{,rr} + u_{,r}$$

for the interval $0 \leq r \leq 1$ with the boundary conditions $u(1) = 0$, $u(0)$ finite is the previously determined kernel $K_0(r, r')$ [7].

This gives us the differential operator relating to our problem. It is the so-called Bessel differential operator L_0, and we now apply it to the integral equation. This yields the zeroth-order Bessel differential equation

$$r^2 \psi_{0,rr} + r \psi_{0,r} + \alpha^2 r^2 \psi_0 = -\lambda_0 r^2 \tag{24a}$$

together with the boundary conditions

$\psi_0(0)$ finite

and

$$\psi_0(1) = \alpha^2 \left(\ln \frac{8}{\varepsilon} - 2\right) \int_0^1 \psi_0(r')\, r'\, dr' + \frac{1}{2} \lambda_0 \left(\ln \frac{8}{\varepsilon} - 2\right) + b_0 + a_0 . \tag{24b}$$

The general homogeneous solutions are the zeroth-order Bessel and Neumann (often called Weber) functions of the first kind. The Neumann functions are not bounded in the origin, which leaves just the zeroth-order Bessel function of the first kind as homogeneous solution:

$$\psi_0^{\mathrm{ah}}(r) = c_0 J_0(\alpha r)\,.$$

A special inhomogeneous solution is immediately obtained with

$$\psi_0^{\mathrm{si}}(s) = -\frac{\lambda_0}{\alpha^2}\,.$$

The general inhomogeneous solution is the sum of the general homogeneous and the special inhomogeneous solution:

$$\psi_0(r) = c_0 J_0(\alpha r) - \frac{\lambda_0}{\alpha^2}\,. \tag{25}$$

To be a solution of the integral equation, this function has to satisfy the boundary condition (24 b). In addition, we impose the requirements of the normalization and subsidiary conditions (22 b), (22 c).

To keep the calculation effort small, first we consider the normalization condition (B 9). From it we can calculate c_0:

$$1 = -2\pi c_0 J_{0,r}(\alpha)\,.$$

With $J_{0,r}(\alpha r) = \alpha J_{0,\alpha r}(\alpha\, r) = -\alpha J_1(\alpha r)$ one obtains

$$c_0 = \frac{1}{2\pi \alpha J_1(\alpha)}\,. \tag{26}$$

Through the subsidiary condition $\psi_0(1) = 0$ we can express λ_0 as a function of α:

$$\lambda_0 = \frac{\alpha J_0(\alpha)}{2\pi J_1(\alpha)}\,, \tag{27}$$

which for small α yields $\lambda_0 = \frac{1}{\pi}(1 - \frac{1}{8}\alpha^2 + O(\alpha^4))$.

As $\psi_0(r)$ is now known for positive α^2, we can investigate the circumstances under which the flux function is monotonic.

For this purpose the argument of the zeroth-order Bessel function must be smaller than the value at which the first minimum occur for $r = 1$. Because of the relation $J_{0,r}(\alpha r) = -\alpha J_1(\alpha r)$ this limit is the first zero j_{11} of the first-order Bessel function:

$$\alpha^2 < j_{11}^2\,.$$

Subject to this constraint, the monotonicity of $\psi_0(r)$ in the interior of the plasma is ensured.

We now insert the results for c_0 and λ_0 in the boundary condition (24 b) and obtain a relation between the logarithm of the aspect ratio and the constants a_0 and b_0:

$$\ln\frac{8}{\varepsilon} - 2 = -2\pi(a_0 + b_0)\,. \tag{28}$$

In addition, the original integral equation (23) simplifies to

$$\psi_0(r) = \alpha^2 \int_0^1 K_0(r, r')\,\psi_0(r')\,r'\,\mathrm{d}r' + \lambda_0 \int_0^1 K_0(r, r')\,r'\,\mathrm{d}r'\,. \tag{29}$$

We now consider the case $\alpha^2 < 0$. We assume α to be purely imaginary and denote the magnitude of α as $|\alpha|$. The solutions for $\alpha^2 < 0$ are obtained through the relation

$$J_n(is) = i^n I_n(s) \tag{30}$$

from that for $\alpha^2 > 0$:

$$\alpha = i\,|\alpha| \quad \text{and} \quad \alpha^2 = -|\alpha|^2 < 0\,. \tag{31}$$

Owing to the monotonicity of $I_0(\alpha r)$ there is no bound for negative α^2; α^2 can be taken to $-\infty$ without problems.

Figure 1 shows λ_0 as a function of $\pm|\alpha|$.

In the case $\alpha = 0$ the integral equation (23) reduces to

$$\psi_0(r) = \frac{\lambda_0}{4}(1 - r^2) + \frac{1}{2}\lambda_0\left(\ln\frac{8}{\varepsilon} - 2\right) + b_0 + a_0\,. \tag{31}$$

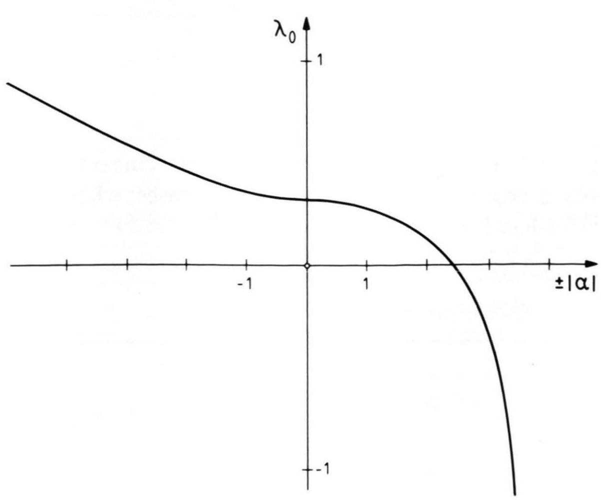

Fig. 1. λ_0 as function of $\pm|\alpha|$. The pole is located at $\alpha = j_{11}$.

The normalization condition (B9) then yields

$$\lambda_0 = \frac{1}{\pi}, \tag{32}$$

and from the subsidiary condition $\psi_0(1) = 0$ we again obtain the relation

$$\ln \frac{8}{\varepsilon} - 2 = -2\pi(a_0 + b_0). \tag{33}$$

The solution for vanishing α is thus

$$\psi_0(r) = \frac{1}{4\pi}(1 - r^2). \tag{34}$$

The results obtained for $\alpha^2 > 0$ and $\alpha^2 < 0$ continuously tend to that in (34).

2.1.2. Various Current Density Profiles

As we have seen, the solution for the flux function $\psi_0(r)$ is continuously dependent on the parameter α^2, and α^2 can take any value in the interval $]-\infty; j_{11}^2[$. Particularly interesting is the effect of α^2 on the current density distribution in the plasma. The introduction of this parameter makes it possible to describe many different current density profiles and take phenomena such as current reversal and surface current into consideration. The current density function in leading order – see (17a) – is

$$-j_{\varphi 0}(r) = \alpha^2 \psi_0(r) + \lambda_0. \tag{35}$$

For positive α^2 we can substitute (27) for λ_0 to obtain

$$-j_{\varphi 0}(r) = -\frac{\alpha J_0(\alpha r)}{2\pi J_1(\alpha)}. \tag{36}$$

Depending on the choice of α^2, various current density profiles are obtained: If $0 \le \alpha^2 \le j_{01}^2$, the current density decreases monotonically as r increases; when the right-hand equality sign is valid, it vanishes at the plasma edge. If $\alpha^2 > j_{01}^2$, the current density changes sign at $r = r_u = j_{01}/\alpha$. For $r > r_u$ the current flows in the opposite direction to that on the magnetic axis; the current reverses. Owing to the constraint $\alpha^2 < j_{11}^2$, which had to be imposed to ensure monotonic $\psi_0(r)$, and which also leads to monotonicity of the current density in the case of linear profile functions, there exists a minimum r for current reversal:

$$r_u > r_{\min}.$$

Current reversal can occur at

$$r_{\min} = \frac{j_{01}}{j_{11}} \cong \frac{2{,}405}{3{,}832} \cong 0{,}628$$

at the earliest. The plasma edge is located at $r = 1$.

If α tends to zero, the current density profile becomes increasingly flat, finally becoming constant throughout the plasma for $\alpha = 0$:

$$j_{\varphi 0}(r) = -\frac{1}{\pi}. \tag{37}$$

In the case $\alpha^2 < 0$ the current density satisfies the equation

$$j_{\varphi 0}(r) = -\frac{|\alpha|\, I_0(|\alpha|\, r)}{2\pi I_1(|\alpha|)}. \tag{38}$$

$I_0(\alpha r)$ has no zero and increases monotonically. The farer α^2 is from zero, the more strongly concentrated is the current at the plasma edge. For large arguments the following asymptotic formula is valid independently of the order n:

$$I_n(s) = (2\pi s)^{-1/2}\, e^s \left[1 + O\left(\frac{1}{s}\right)\right],$$

and so the current density can be expressed as

$$j_{\varphi 0}(r) \xrightarrow[|\alpha| \to \infty]{} -\frac{|\alpha|}{2\pi} r^{-1/2}\, e^{|\alpha|(r-1)}. \tag{39}$$

For $r \in]0; 1[$ the exponent is negative and the current density vanishes when α grows beyond all limits.

In the limiting case $r = 0$ the exponential function dominates the pole of $r^{-1/2}$ and $j_{\varphi 0}$ likewise vanishes.

Finally, at the plasma edge, the exponent becomes zero and the current density goes linearly in $|\alpha|$ to infinity:

$$j_{\varphi 0}(1) \xrightarrow[|\alpha| \to \infty]{} -\frac{|\alpha|}{2\pi}.$$

In the limiting case $\alpha^2 \to -\infty$ the entire current is concentrated in an infinitely thin region at the plasma edge: The normalization of the total current to one is retained.

Figure 2 shows $-j_\varphi(r)$ for a few values of α.

2.2. *First Order*

2.2.1. The Ansatz

Taking into account that the upper limit of the r'-integration is a function of ε and θ', one obtains for

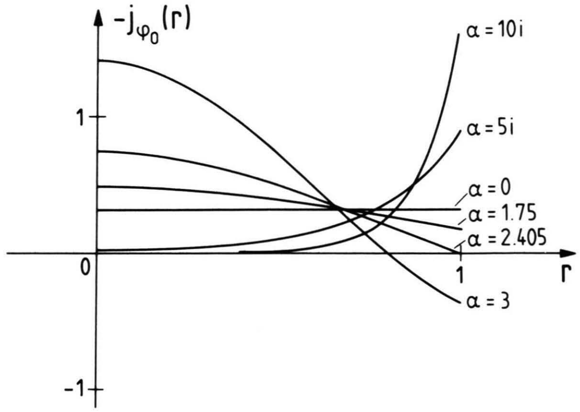

Fig. 2. Toroidal current density profiles for typical values of α. $r = 0$: magnetic axis, $r = 1$: plasma edge.

the first order the integral equation

$$\begin{aligned}\psi_1(r,\theta) = &-\int_0^{2\pi} \hat{r}_1(\theta')\, G_0(r,\theta,1,\theta')\, j_{\varphi 0}(1,\theta')\,\mathrm{d}\theta' \\ &-\int_0^{2\pi}\int_0^1 G_0(r,\theta,r',\theta')\, j_{\varphi 1}(r',\theta')\, r'\,\mathrm{d}r'\,\mathrm{d}\theta' \\ &-\int_0^{2\pi}\int_0^1 G_1(r,\theta,r',\theta')\, j_{\varphi 0}(r',\theta')\, r'\,\mathrm{d}r'\,\mathrm{d}\theta' \\ &+ a_1 + b_1 + 2 b_0\, r\cos\theta\,. \end{aligned} \tag{40}$$

We substitute in the integrand the results of the expansion of Green's function (18) and the current density (19) and do the θ'-integration except for the term containing ψ_1.

The inhomogeneity splits into a θ-independent component, terms having a factor $\cos\theta$, and an expression with $\hat{r}_1(\theta)$, whose θ-dependence is uncertain. The structure of the inhomogeneity suggests for $\psi_1(r,\theta)$ the ansatz

$$\psi_1(r,\theta) = \psi_{10}(r) + \psi_{11}(r)\cos\theta\,. \tag{41}$$

It then follows from the expansion of the edge curve (A9) that $\hat{r}_1(\theta)$ also takes the form of (41):

$$\begin{aligned}\hat{r}_1(\theta) &= -\frac{\psi_1(1,\theta)}{\psi_{0,r}(1)} = -\frac{\psi_{10}(1)}{\psi_{0,r}(1)} - \frac{\psi_{11}(1)}{\psi_{0,r}(1)}\cos\theta \\ &= \hat{r}_{10} + \hat{r}_{11}\cos\theta\,. \end{aligned} \tag{42}$$

The subsidiary condition (21 c) yields

$$\psi_{10}(1) = \psi_{11}(1) = 0\,. \tag{43}$$

According to (42) this means that the correction of the edge curve $\hat{r}_1(\theta)$ vanishes in first order:

$$\hat{r}_1(\theta) \equiv 0\,. \tag{44}$$

Deviation of the plasma edge from the circular cross-section is thus only to be expected as of second order. The calculation yields the two integral equations

$$\begin{aligned}\psi_{10}(r) = &\ \alpha^2 \int K_0(r,r')\,\psi_{10}(r')\, r'\,\mathrm{d}r' \\ &+ \alpha^2\left(\ln\frac{8}{\varepsilon} - 2\right)\int \psi_{10}(r')\, r'\,\mathrm{d}r' \\ &+ \frac{\lambda_1}{2}\left(\ln\frac{8}{\varepsilon} - 2\right) + \lambda_1 \int K_0(r,r')\, r'\,\mathrm{d}r' + a_1 + b_1\,, \end{aligned} \tag{45 a}$$

$$\begin{aligned}\psi_{11}(r) = &\ \alpha^2 \int K_1(r,r')\,\psi_{11}(r')\, r'\,\mathrm{d}r' \\ &+ 2\left(\beta_P - \tfrac{1}{4}\right)\alpha^2 \int K_1(r,r')\,\psi_0(r')\, r'^2\,\mathrm{d}r' \\ &+ \left(2\nu\beta_P^2 + 2(\lambda_0 - \nu)\,\beta_P - \frac{\lambda_0}{2}\right)\int K_1(r,r')\, r'^2\,\mathrm{d}r' \\ &+ \frac{\alpha^2}{2}\left(\ln\frac{8}{\varepsilon} - 1\right) r \int \psi_0(r')\, r'\,\mathrm{d}r' \\ &+ \frac{\alpha^2}{2}\, r \int K_0(r,r')\,\psi_0(r')\, r'\,\mathrm{d}r' \\ &+ \frac{\lambda_0}{2}\, r \int K_0(r,r')\, r'\,\mathrm{d}r' + \left[2 b_0 + \frac{\lambda_0}{4}\left(\ln\frac{8}{\varepsilon} - 1\right)\right] r\,. \end{aligned} \tag{45 b}$$

We have used the abbreviation

$$K_1(r,r') = \begin{cases} \dfrac{1}{2}\dfrac{r}{r'}\,, & r \le r'\,, \\[2mm] \dfrac{1}{2}\dfrac{r'}{r}\,, & r \ge r'\,, \end{cases}$$

which corresponds to

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\frac{1}{\varrho}\cos n\theta'\,\mathrm{d}\theta' = K_n(r,r')\cos n\theta$$

with

$$K_n(r,r') = \frac{1}{2^n}\begin{cases} \left(\dfrac{r}{r'}\right)^n, & r \le r' \\[2mm] \left(\dfrac{r'}{r}\right)^n, & r \ge r' \end{cases}\,, \quad n = 1, 2, 3, \dots\,.$$

The solutions of these integral equations have to satisfy the normalization and subsidiary conditions

(N) $\quad \psi_{10,r}(1) = 0\,,$ (45 c)

(B) $\quad \psi_{10}(1) = \psi_{11}(1) = 0\,.$ (45 d)

Investigation of the θ-independent component shows that $\psi_{10}(r)$ vanishes identically for all $\alpha^2 \in]-\infty; j_{11}^2[$:

$$\psi_{10}(r) \equiv 0\,. \tag{46}$$

In addition it holds that

$$a_1 + b_1 = 0, \quad \lambda_1 = 0\,. \tag{47}$$

Analysis of the θ-dependent component yields the parameter b_0, the strength of the dimensionless homogeneous vertical field in leading order, and a_0:

$$b_0 = -\frac{1}{8\pi}\left(\ln\frac{8}{\varepsilon} - 1 + \beta_P\right) + \frac{1-\pi\lambda_0}{\alpha^2}\left[-\nu\beta_P^2 + \nu\beta_P + \tfrac{1}{4}\lambda_0\right]. \tag{48a}$$

$$a_0 = -\frac{1}{8\pi}\left(3\ln\frac{8}{\varepsilon} - 7 - \beta_P\right) - \frac{1-\pi\lambda_0}{\alpha^2}\left[-\nu\beta_P^2 + \nu\beta_P + \tfrac{1}{4}\lambda_0\right]. \tag{48b}$$

We now have the complete solution of the flux function up to and including first order:

$$\psi(r,\theta) = \psi_0(r) + \varepsilon\psi_1(r,\theta) + O(\varepsilon^2) \tag{49a}$$

with

$$\psi_0(r) = c_0 J_0(\alpha r) - \frac{\lambda_0}{\alpha^2}, \tag{49b}$$

$$\psi_1(r,\theta) = \psi_{11}(r)\cos\theta\,, \tag{49c}$$

and

$$\psi_{11}(r) = \left\{-\frac{2}{\alpha^2}\nu r + \frac{4\pi c_0 \nu}{\alpha} J_1(\alpha r)\right\}\beta_P^2 + \left\{\frac{2}{\alpha^2}\nu r + \left[\frac{c_0}{2}\alpha(1-r^2) - \frac{4\pi c_0 \nu}{\alpha}\right] J_1(\alpha r)\right\}\beta_P + \frac{c_0}{2} r J_0(\alpha r) - \frac{\pi c_0 \lambda_0}{\alpha} J_1(\alpha r)\,. \tag{49d}$$

For $\alpha \to 0$ (49d) reduces to

$$\psi_{11}(r) = \frac{1}{4}\left[\nu\beta_P^2 + \left(\frac{1}{\pi} - \nu\right)\beta_P + \frac{1}{4\pi}\right](r - r^3)\,.$$

The results of the flux function in first order (49) and the homogeneous vertical field (48a) were already published in 1963 by Shafranov [9]. Unlike Shafranov's study, the present paper uses an integral equation method and evaluates to second order. In principle, however, this expansion can be taken self-consistently to any order.

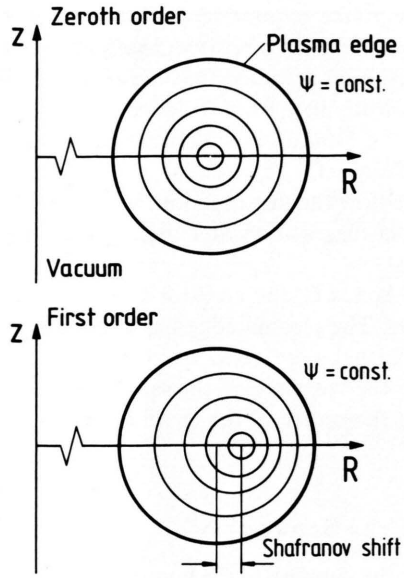

Fig. 3. Shafranov shift. Shown are the curves ψ = const in zeroth and first order. The plasma edge remains unperturbed.

2.2.2. Shafranov Shift and Contour Lines

The position of the magnetic axis of the plasma is of physical interest. It is characterized by the vanishing of the gradient of the flux function:

$$\nabla\psi(r_s, \theta_s) = 0\,. \tag{50}$$

The extremum of the flux function is on the x-axis. We make the ansatz

$$x_s = x_{s0} + \varepsilon x_{s1} + O(\varepsilon^2)\,. \tag{51}$$

In this way we expand (49a) at the location $x = x_s$, $y = 0$ and obtain in zeroth order

$$0 = c_0 J_{0,x}(\alpha x_{s0})\,,$$

which, owing to $\alpha^2 < j_{11}^2$, is equivalent to

$$x_{s0} = 0\,. \tag{52}$$

The first order arranged in powers of β_P, yields

$$x_{s1} = \frac{4\nu}{\alpha^2}\left(\pi - \frac{1}{c_0\alpha^2}\right)\beta_P^2 + \left[\frac{1}{2} - \frac{4\nu}{\alpha^2}\left(\pi - \frac{1}{c_0\alpha^2}\right)\right]\beta_P + \frac{1-\pi\lambda_0}{\alpha^2}\,. \tag{53}$$

For $\alpha^2 > 0$ and positive pressure it is then possible to show positive definiteness of x_{s1}.

With negative α^2 the requirement of positive pressure is no longer sufficient for positive Shafranov shift. Only when it has been assumed that the pressure increases monotonically in ψ from the edge to the magnetic axis does the Shafranov shift become positive. For details see Annex C.

In order to see how the flux function behaves in the interior of the plasma, we consider the contour lines $\psi(r, \theta) = c = \text{const.}$

As shown in Annex C, the contour lines are non-concentric circles. The plasma edge has the origin as its centre. The farther the circles are away from the edge, the smaller they become; at the same time, however, the shift of the centre of the circle increases.

2.2.3. Form of the Separatrix

Knowledge of the strength of the homogeneous vertical field provides qualitative data on the flux function in vacuum. Attention can be restricted to its behaviour in the immediate vicinity of the torus axis. That is where the shape of the separatrix is determined. The intersection of a magnetic surface with a poloidal plane generates a curve which is called a separatrix when at least one so-called X-point exists on it. At the X-point the poloidal magnetic field vanishes and, consequently, the gradient of the flux function as well:

$$\nabla\psi(R, z) = 0 \,. \tag{54}$$

In principle, two kinds of separatrices are conceivable in our configuration. The intersections with a poloidal surface enclose the plasma either in a D-shape or in the form of a (drop-shaped) loop. The various geometries are sketched in Figs. 4 and 5.

In order to decide now what kind of separatrix is present, it is sufficient to look at the flux function on the R-axis ($z = 0$) for small R. In the drop-shaped case there is an X-point between the magnetic axis (itself an O-point) and the torus axis, i.e. in this R-interval there is a minimum of $\psi(R, 0)$. For small R, $\partial\psi/\partial R$ is then negative, in the D-shaped case it is positive.

In order to obtain an expression for the flux function in the immediate vicinity of the torus axis, we think of the plasma cross-section as being concentrated at the point (1, 0). One then has

$$\psi(R, z) = G(R, 1, z, 0) + bR^2 + a \,,$$

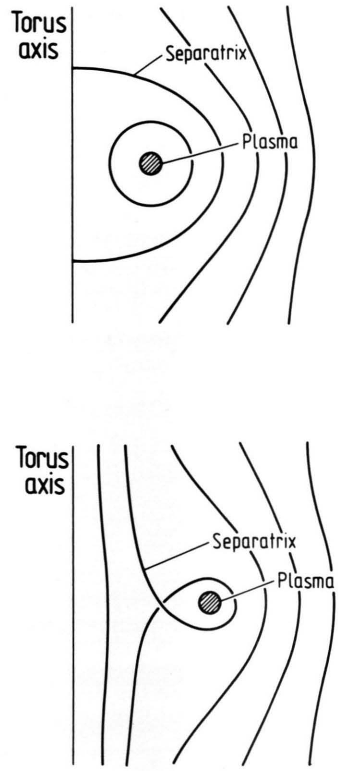

Fig. 4. D- and drop-shaped separatrices (qualitative).

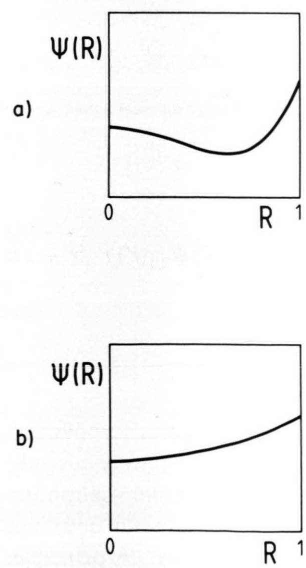

Fig. 5. Flux functions (qualitative) for D- and drop-shaped separatrices. Torus axis at $R = 0$, plasma at $R = 1$.

and the module of the elliptic integrals in $G(R,z,R',z')$ is

$$k^2 = \frac{4R}{(1+R)^2},$$

where $R = 1 + \varepsilon x$.

We take into account the fact that we normalized the total current to minus one, and expand the elliptic integrals for small R. This then yields

$$\psi(R, 0) = (b + \tfrac{1}{4}) R^2 + a\,. \tag{55}$$

a is always negative, and we obtain a minimum (X-point) for the case $b < -1/4$. The separatrix is then drop-shaped. If $b > -1/4$, it is D-shaped.

2.3. Second Order

2.3.1. The Ansatz

We now consider the second order of the integral equations (21):

$$\psi(r, \theta) = -\int_0^{2\pi}\int_0^{\hat{r}(\theta')} G(r, r', \theta, \theta')\, j_\varphi(r', \theta')\, r'\, dr'\, d\theta' + b(1 + \varepsilon r \cos\theta)^2 + a\,.$$

According to the results in first order one has

$$\begin{aligned}\psi_2(r, \theta) = &-\int_0^{2\pi} \hat{r}_2(\theta')\, G_0(r, \theta, 1, \theta')\, j_{\varphi 0}(1, \theta')\, d\theta' \\ &-\int_0^{2\pi}\int_0^1 G_0(r, \theta, r', \theta')\, j_{\varphi 2}(r', \theta')\, r'\, dr'\, d\theta' \\ &-\int_0^{2\pi}\int_0^1 G_2(r, \theta, r', \theta')\, j_{\varphi 0}(r', \theta')\, r'\, dr'\, d\theta' \\ &-\int_0^{2\pi}\int_0^1 G_1(r, \theta, r', \theta')\, j_{\varphi 1}(r', \theta')\, r'\, dr'\, d\theta' \\ &+ a_2 + b_2 + 2 b_1 r\cos\theta + b_0 r^2 \cos^2\theta\,. \end{aligned} \tag{56a}$$

The solution has to satisfy the normalization condition (B12)

$$\psi_{20,r}(1) - 2\pi\lambda_0 \psi_{20}(1) = \tfrac{1}{2}(\psi_{11,r}(1) - \psi_{0,r}(1)) \tag{56b}$$

and the subsidiary condition

$$\text{(B)} \quad \psi_2(1, 0) = \psi_2(1, \pi) = 0\,. \tag{56c}$$

According to (A11) and (B9) one has

$$\hat{r}_2(\theta) = -\frac{\psi_2(1, \theta)}{\psi_{0,r}(1)} = 2\pi\,\psi_2(1, \theta)\,, \tag{57}$$

so that the first term in (56a) must be regarded as homogeneous. We substitute in the integrand the results of the expansion of the current density (19) and Green's function and can then do the θ'-integration except for the term containing $\psi_2(r, \theta)$.

We use the abbreviation

$$K_2(r, r') = \frac{1}{4}\begin{cases}\left(\dfrac{r}{r'}\right)^2, & r \le r' \\[2ex] \left(\dfrac{r'}{r}\right)^2, & r \ge r'\,.\end{cases}$$

It is found that the inhomogeneity can be completely decomposed into a $\cos 2\theta$-dependent, a $\cos\theta$-dependent, and a θ-independent component. By analogy with the discussion in first order, we make the ansatz

$$\psi_2(r, \theta) = \psi_{20}(r) + \psi_{21}(r)\cos\theta + \psi_{22}(r)\cos 2\theta\,. \tag{58}$$

This is put into the integral equation (56a), and in this way we get three defining equations for $\psi_{20}(r)$, $\psi_{21}(r)$, and $\psi_{22}(r)$:

$$\begin{aligned}\psi_{20}(r) = {}&\alpha^2 \int\left(\ln\frac{8}{\varepsilon} - 2 + K_0(r, r')\right)\psi_{20}(r')\, r'\, dr' + 2\pi\lambda_0\left(\ln\frac{8}{\varepsilon} - 2\right)\psi_{20}(1) + \lambda_2 \int\left(\ln\frac{8}{\varepsilon} - 2 + K_0(r, r')\right) r'\, dr' \\ &+ \frac{1}{2}\int[(2\beta_P - 1)\,\alpha^2 r'\,\psi_{11}(r') + (1 - \beta_P)(\alpha^2\psi_0(r') + \lambda_0 + \beta_P \nu) r'^2]\left[\ln\frac{8}{\varepsilon} - 2 + K_0(r, r')\right] r'\, dr' \\ &+ \int(\alpha^2\psi_0(r') + \lambda_0)\left[\frac{1}{16}(r^2 + r'^2)\left(2\ln\frac{8}{\varepsilon} + 1 + 2K_0(r, r')\right) - \frac{1}{4} r r' K_1(r, r')\right] r'\, dr' \\ &+ \frac{1}{4}\int\{\alpha^2\psi_{11}(r') + r'\,[(2\beta_P - 1)(\alpha^2\psi_0(r') + \lambda_0) + 2\beta_P(\beta_P - 1)\nu]\} \\ &\cdot\left\{\left(\ln\frac{8}{\varepsilon} - 1\right) r' + r' K_0(r, r') + r K_1(r, r')\right\} r'\, dr' + a_2 + b_2 + \frac{b_0}{2} r^2\,, \end{aligned} \tag{59a}$$

$$\psi_{21}(r) = \alpha^2 \int K_1(r, r')\, \psi_{21}(r')\, r'\, \mathrm{d}r' + \pi \lambda_0 \psi_{21}(1) r + 2 b_1 r\,, \tag{59 b}$$

and

$$\begin{aligned}\psi_{22}(r) = {} & \alpha^2 \int K_2(r, r')\, \psi_{22}(r')\, r'\, \mathrm{d}r' + \frac{\pi}{2} \lambda_0\, \psi_{22}(1)\, r^2 \\ & + \frac{1}{2} \int K_2(r, r') \left[(2\beta_\mathrm{P} - 1)\, \alpha^2 r'\, \psi_{11}(r') + (1 - \beta_\mathrm{P})\left(\alpha^2 \psi_0(r') + \lambda_0 + \beta_\mathrm{P} v\right) r'^2\right] r'\, \mathrm{d}r' \\ & + \frac{1}{16} \int \left(\alpha^2 \psi_0(r') + \lambda_0\right) \left[-\left(\ln \frac{8}{\varepsilon} - 2\right) r^2 - r^2 K_0(r, r') + 2 r r' K_1(r, r') - r'^2 K_2(r, r')\right] r'\, \mathrm{d}r' \\ & + \frac{1}{4} \int \left\{\alpha^2 \psi_{11}(r') + r'\left[(2\beta_\mathrm{P} - 1)\left(\alpha^2 \psi_0(r') + \lambda_0\right) + 2\beta_\mathrm{P}(\beta_\mathrm{P} - 1) v\right]\right\} \left\{r K_1(r, r') + r' K_2(r, r')\right\} r'\, \mathrm{d}r' + \frac{b_0}{2} r^2 .\end{aligned} \tag{59 c}$$

2.3.2. The Unique Solvability of the Inhomogeneous Problem

Before we set about solving the inhomogeneous integral equations (59) for $\psi_2(r, \theta)$, let us see what requirements are imposed on the solutions by the normalization condition and the subsidiary condition, or whether these requirements can be met at all. First we set out all the conditions:

$$\text{(N)}\quad \psi_{20,r}(1) - 2\pi \lambda_0 \psi_{20}(1) = \tfrac{1}{2}\left(\psi_{11,r}(1) - \psi_{0,r}(1)\right), \tag{60 a}$$

$$\text{(B)}\quad \psi_{20}(1) + \psi_{21}(1) + \psi_{22}(1) = 0 \quad \text{for } \theta = 0, \tag{60 b}$$

$$\text{(B)}\quad \psi_{20}(1) - \psi_{21}(1) + \psi_{22}(1) = 0 \quad \text{for } \theta = \pi. \tag{60 c}$$

These are three linear equations for our four unknowns still occurring in second order, λ_2, a_2, b_2, and b_1.

b_0 could not be calculated in zeroth order but only as of first order, and so it is expected that b_{n-1} generally cannot be calculated until n-th order. The reason is that in n-th order the subsidiary condition is of the form

$$a_n + b_n + 2 b_{n-1} = \dots ,$$
$$a_n + b_n - 2 b_{n-1} = \dots .$$

The determinant of the solution vector (a_n, b_{n-1}) does not vanish, and we remove b_n from the set of unknowns that can be determined in n-th order, since it represents a result of the next-higher order. We are then left with a system of three equations for three unknowns which can be transformed as follows:

$$\text{(N)}\quad \psi_{20,r}(1) - 2\pi \lambda_0 \psi_{20}(1) = \tfrac{1}{2}\left(\psi_{11,r}(1) - \psi_{0,r}(1)\right), \tag{61 a}$$

$$(\text{B}_1)\quad \psi_{20}(1) + \psi_{22}(1) = 0, \tag{61 b}$$

$$(\text{B}_2)\quad \psi_{21}(1) = 0 . \tag{61 c}$$

As $\psi_{20}(r)$ depends on λ_2 and a_2, $\psi_{21}(r)$ depends only on b_1 and $\psi_{22}(r)$ is completely determined, we have the following system of linear equations in matrix representation:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} \lambda_2 \\ a_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{62}$$

or, in short,

$$A \boldsymbol{x} = \boldsymbol{f} .$$

Here we are not at all interested in the explicit solution vector; rather it is sufficient for us to show that a unique solution exists, i.e. the determinant does not vanish. In the first part of Annex D it is shown that λ_2, a_2, and b_1 can always be chosen such that they satisfy all conditions imposed on $\psi_2(r, \theta)$.

In considering the flux function in second order we shall see that it is not necessary for our purpose to know ψ_{20}, ψ_{21}, and ψ_{22}.

Analysis of $\psi_{10}(r)$ yields

$$\psi_{21}(r) \equiv 0, \quad b_1 = 0 . \tag{63}$$

The component of $\psi_2(r, \theta)$ that varies as $\cos\theta$ thus vanishes identically, and the component for $\psi_2(r, \theta)$ simplifies to

$$\psi_2(r, \theta) = \psi_{20}(r) + \psi_{22}(r) \cos 2\theta . \tag{64}$$

2.3.3. The Edge Curve

We are now interested in the shape of the plasma edge. As the edge perturbation vanishes in first order, the edge curve is described by

$$\hat{r}(\theta) = 1 - \varepsilon^2 \frac{\psi_2(1, \theta)}{\psi_{0,r}(1)} = 1 - \varepsilon^2 \frac{\psi_{20}(1) + \psi_{22}(1) \cos 2\theta}{\psi_{0,r}(1)} . \tag{65}$$

The edge curve in second order is an ellipse whose axes coincide with the coordinate axes.

The next question to be asked is what the ratio of the two half-axes $H_R = b/a = b$ is. It is exactly equivalent to the distance between the intersection of the edge curve with the y-axis and the origin of the coordinate system:

$$H_R = \hat{r}\left(\frac{\pi}{2}\right) = 1 - \varepsilon^2 \frac{\psi_{20}(1) - \psi_{22}(1)}{\psi_{0,r}(1)} .$$

Owing to the subsidiary condition (61 b) this reduces to

$$H_R = 1 + 2\varepsilon^2 \frac{\psi_{22}(1)}{\psi_{0,r}(1)} = 1 - 4\pi\varepsilon^2 \psi_{22}(1) . \quad (66)$$

We distinguish three cases:

1. $\psi_{22}(1) < 0$ ellipse with vertical major axis,
2. $\psi_{22}(1) = 0$ circle,
3. $\psi_{22}(1) > 0$ ellipse with horizontal major axis.

To determine the parameters of the ellipse, one need not calculate the whole function $\psi_2(r, \theta)$; it is sufficient to know $\psi_{22}(1)$.

2.3.4. Half-Axis Ratio Near the Magnetic Axis

By analogy with first order, the contour lines for $c \neq 0$ are expected to be ellipses as well, like the edge curve, but with variable aspect ratio. We are particularly interested in the readily accessible half-axis ratio H_A in the immediate vicinity of the magnetic axis.

Since the maximum of ψ is only at a distance $O(\varepsilon)$ from the origin, the vicinity of the maximum of ψ can be described by a Taylor expansion in r (neglecting terms of $O(r^3) \sim O(\varepsilon^3)$):

$$\begin{aligned}\psi &= \psi_0(r) + \varepsilon\psi_{11}(r)\cos\theta \\ &\quad + \varepsilon^2[\psi_{20}(r) + \psi_{22}(r)\cos 2\theta] + O(\varepsilon^3) \quad (67)\\ &= \psi_{00} + \psi_{02} r^2 + \varepsilon\psi_{111} r\cos\theta \\ &\quad + \varepsilon^2[\psi_{200} + \psi_{202} r^2 + \psi_{222} r^2 \cos 2\theta] + O(\varepsilon^3) \\ &= \psi_{00} + \psi_{02}(x^2+y^2) + \varepsilon\psi_{111} x \\ &\quad + \varepsilon^3[\psi_{200} + \psi_{202}(x^2+y^2) + \psi_{222}(x^2-y^2)] + O(\varepsilon^3) .\end{aligned}$$

Now, the maximum of ψ is situated at $y = y_0 = 0$ and

$$\begin{aligned}x = x_0 &= -\tfrac{1}{2}\varepsilon\psi_{111}[\psi_{02} + \varepsilon^2(\psi_{202} + \psi_{222})]^{-1} \\ &= -\frac{\varepsilon\psi_{111}}{2\psi_{02}} + O(\varepsilon^3) . \quad (68)\end{aligned}$$

Thus, the contour lines of ψ are, in leading order, described by

$$\begin{aligned}&(x-x_0)^2[\psi_{02} + \varepsilon^2(\psi_{202} + \psi_{222})] \\ &\quad + y^2[\psi_{02} + \varepsilon^2(\psi_{202} - \psi_{222})] = \text{const} , \quad (69)\end{aligned}$$

which are ellipses with half-axis ratio

$$\begin{aligned}H_A &= \frac{[\psi_{02} + \varepsilon^2(\psi_{202} + \psi_{222})]^{1/2}}{[\psi_{02} + \varepsilon^2(\psi_{202} - \psi_{222})]^{1/2}} \quad (70)\\ &= 1 + \varepsilon^2 \frac{\psi_{222}}{\psi_{02}} + O(\varepsilon^4) = 1 + \varepsilon^2 \frac{\psi_{22,rr}(0)}{\psi_{0,rr}(0)} + O(\varepsilon^4) .\end{aligned}$$

So, for the half-axis ratio on the magnetic axis we do not need the whole function $\psi_2(r, \theta)$ but only the component $\psi_{22,rr}(0)$.

2.3.5. Results and Special Cases

The integral equation (59 c) for $\psi_{22}(r)$ is solved in the second half of Annex D. This enables us to discuss the form of the flux function qualitatively. In particular, we shall give the results for the half-axis ratios of the flux function at the plasma edge H_R and on the magnetic axis H_A. All numerical results given are for $\varepsilon = 0.1$. For H_R we obtained

$$H_R = 1 - 4\pi\varepsilon^2\psi_{22}(1) , \quad (71\,a)$$

and now we know $\psi_{22}(1)$:

$$\begin{aligned}\psi_{22}(1) &= \frac{1 - \pi\lambda_0}{\pi c_0 \alpha^2}[\beta_P^3 c_{223} + \beta_P^2 c_{222} + \beta_P c_{221} + c_{220}] \\ &\quad + \beta_P^3\left[\frac{\pi\lambda_0 \nu}{\alpha^2} + 2\frac{\nu}{\alpha^2}\right] \\ &\quad + \beta_P^2\left[\frac{1}{16\pi}(1 + \pi\lambda_0) - \frac{\nu}{\alpha^2}(2 + \pi\lambda_0)\right] \\ &\quad + \beta_P\left[-\frac{\pi\lambda_0^2}{4\alpha^2} - \frac{1}{24\pi}\right] - \frac{3\lambda_0}{16\alpha^2} . \quad (71\,b)\end{aligned}$$

H_A has likewise been calcuated:

$$H_A = 1 + \varepsilon^2 \frac{\psi_{22,rr}(0)}{\psi_{0,rr}(0)} \quad (72\,a)$$

with

$$\begin{aligned}\psi_{22,rr}(0) &= \frac{\alpha^2}{4}(c_{223}\beta^3 + c_{222}\beta^2 + c_{221}\beta + c_{220}) \\ &\quad + \beta^3\left(2\pi c_0 \nu + \frac{4\nu}{\alpha^2}\right) + \beta^2\left(\frac{c_0\alpha^2}{4} - \pi c_0 \nu - \frac{5\nu}{\alpha^2}\right)\end{aligned}$$

$$+\beta\left(-\frac{\pi c_0 \lambda_0}{2}+\frac{c_0 \alpha^2}{8}-\pi c_0 v+\frac{v}{\alpha^2}\right)$$

$$-\frac{c_0}{8}-\frac{\pi c_0 \lambda_0}{4}. \tag{72a}$$

Let us now discuss these results in the case of two special profiles. On the one hand, we shall assume that both the toroidal and poloidal current densities and the pressure gradient at the plasma edge vanish. This corresponds to the following choices of parameters:

$$\alpha^2 = j_{01}^2, \tag{73a}$$

$$v = 0. \tag{73b}$$

Consequently, both λ_0 and ξ and η vanish according to (13) and (27). For the edge ellipses we then obtain

$$H_R = 1+\frac{\varepsilon^2}{\alpha^2}\left\{\frac{1}{2}(4-\alpha^2)\,\beta_P^2+\frac{1}{6}(\alpha^2-8)\,\beta_P + 3\ln\frac{8}{\varepsilon}-4\right\}. \tag{74}$$

The term in braces in (74) is positive for small β_P down to its zero at about $\beta_P=3$. The edge curve in each case is an ellipse with vertical major half-axis. H_R is given for some typical values of the poloidal beta and for $\varepsilon = 0.1$:

β_P	H_R
0	1.016
0.5	1.015
1	1.014
1.5	1.011

At $\beta_P=3$ the edge curve is a circle, for higher β_p an ellipse with horizontal major axis.

The half-axis ratio on the magnetic axis is given by the expression

$$H_A = 1+\varepsilon^2\left\{-\left(\frac{1}{4}+\frac{\alpha^2}{32}\right)\beta_P^2-\frac{5}{12}\beta_P+\frac{1}{4\alpha^2} + \frac{1}{8}\left(3\ln\frac{8}{\varepsilon}-4\right)\right\}. \tag{75}$$

For $\varepsilon=0.1$ the following values are calculated:

β_P	H_A
0	1.012
0.5	1.009
1	1.003
1.5	0.996

Next we want to see what results are obtained for a flat toroidal current profile. This corresponds to the choice $\alpha = 0$.

We calculate

$$H_R = 1+\varepsilon^2\left\{-\frac{1}{16}+\frac{1}{4}\left(3\ln\frac{8}{\varepsilon}-4\right)\right\}. \tag{76}$$

This corresponds to the well-known result [10]. However, for $v(1-\beta_P) \neq 0$ formula (76) contradicts the result in [11] and [12].

H_R is independent of the poloidal beta and has (for $\varepsilon=0.1$) the value

$$H_R = 1.022\,.$$

For the half-axis ratio on the magnetic axis we obtain

$$H_A = 1+\varepsilon^2\left\{-\frac{1}{4}\pi v \beta_P^2+\frac{1}{4}(\pi v-1)\,\beta_P + \frac{1}{4}\left(3\ln\frac{8}{\varepsilon}-4\right)\right\}. \tag{77}$$

For $\varepsilon=0.1$ and $v=0$ the following values are calulated:

β_P	H_A
0	1.023
0.5	1.022
1	1.020
1.5	0.019

Summary

We have considered an axisymmetric MHD equilibrium with an external homogeneous magnetic field which is parallel to the torus axis (axis of symmetry). The magnetic flux function of the equilibrium is described by an integral equation representing a nonlinear free-boundary problem. This is solved by an expansion with respect to the inverse aspect ratio ε. In keeping with a two-scale method the quantities ε and $\ln\varepsilon$ are considered as independent. Linear profile functions containing four parameters are used.

In leading order (ε^0) the flux function does not depend on the poloidal angle, so that the level lines are concentric circles. In first order in ε the plasma surface is unchanged and the level lines are non-concentric circles (Shafranov-Shift). In second order it is found that the plasma surface and the level lines are elliptically deformed. In order to satisfy the solubility conditions, only three of the four profile parameters can be chosen independently. This latter fact has not been correctly treated in references [11] and [12], with the consequence that the formulae describing the ellipticity in second order are different. In a forthcoming paper it will be shown how the profile parameters have to be chosen to make the configuration stable.

Annex A. Expansion of the Edge Curve

The edge of the plasma is described by the condition

$$\psi(r, \theta) = 0 . \tag{A.1}$$

The set of all points (r, θ), satisfying this criterion is called the edge curve and is denoted by $\hat{r}(\theta)$. The edge curve is thus defined by the relation

$$\psi(\hat{r}(\theta), \theta) = 0 . \tag{A.2}$$

In order to see what effect the aspect ratio has on the shape of this curve, we expand $\psi(r, \theta)$ and $\hat{r}(\theta)$ in ε:

$$\psi(r, \theta) = \psi_0(r) + \varepsilon\psi_1(r, \theta) + \varepsilon^2\psi_2(r, \theta) + O(\varepsilon^3) , \tag{A.3}$$

$$\hat{r}(\theta) = \hat{r}_0(\theta) + \varepsilon\hat{r}_1(\theta) + \varepsilon^2\hat{r}_2(\theta) + O(\varepsilon^3) . \tag{A.4}$$

Substituting in (A.2) now yields

$$\begin{aligned} 0 = \psi_0(\hat{r}_0 + -\varepsilon\hat{r}_1 + \varepsilon^2\hat{r}_2 + \ldots) \\ + \varepsilon\psi_1(\hat{r}_0 + \varepsilon\hat{r}_1 + \varepsilon^2\hat{r}_2 + \ldots, \theta) \\ + \varepsilon^2\psi_2(\hat{r}_0 + \varepsilon\hat{r}_1 + \varepsilon^2\hat{r}_2 + \ldots, \theta) + \ldots . \end{aligned} \tag{A.5}$$

Expanding this according to Taylor, we obtain

$$\begin{aligned} 0 = \psi_0(\hat{r}_0) + \varepsilon[\hat{r}_1\psi_{0,r}(\hat{r}_0) + \psi_1(\hat{r}_0)] \\ + \varepsilon^2[\hat{r}_2\psi_{0,r}(\hat{r}_0) + \tfrac{1}{2}\hat{r}_1^2\psi_{0,rr}(\hat{r}_0) + \hat{r}_1\psi_{1,r}(\hat{r}_0) \\ + \psi_2(\hat{r}_0)] + O(\varepsilon^3) . \end{aligned} \tag{A.6}$$

From zeroth order of this equation

$$0 = \psi_0(\hat{r}_0) \tag{A.7}$$

it is seen that $\hat{r}_0$ – since ψ_0 is independent of the poloidal angle – will not be a function of θ. The plasma boundary condition requires that $\psi(1, 0) = \psi(1, \pi) = 0$, thus fixing the value of $\hat{r}_0$:

$$\hat{r}_0 = 1 . \tag{A.8}$$

In leading order the plasma edge is thus a circle with radius 1.

The first order gives

$$\hat{r}_1(\theta) = -\frac{\psi_1(1, \theta)}{\psi_{0,r}(1)} \tag{A.9}$$

and the second order finally yields

$$\hat{r}_2(\theta) = -\frac{\hat{r}_1^2(\theta)\,\psi_{0,rr}(1) + 2\hat{r}_1\psi_{1,r}(1, \theta) + 2\psi_2(1, \theta)}{2\psi_{0,r}(1)} . \tag{A.10}$$

The solution of the integral equation for $\psi_1(r, \theta)$ in the main body of the text yields the result that $\psi_1(1, \theta)$ and hence also the edge perturbation $\hat{r}_1(\theta)$ vanishes in first order for all θ: $\psi_1(1, \theta) = 0$. The expression for the edge curve thus reduces to

$$\hat{r}(\theta) = 1 - \varepsilon^2 \frac{\psi_2(1, \theta)}{\psi_{0,r}(1)} . \tag{A.11}$$

Only as of second order does the plasma edge thus deviate from the circular shape, insofar as $\psi_2(1, \theta)$ is non-zero.

Annex B. Expansion of the Normalization Condition

The normalization condition (14b) is

$$1 = -\int_{\partial D} \frac{\partial\psi}{\partial n}\frac{1}{R}\,\mathrm{d}l . \tag{B.1}$$

To write the expansion, we have to expand $\partial\psi/\partial n$, $1/R$ and $\mathrm{d}l$ up to and including second order in the inverse aspect ratio. First we make use of the relation

$$\frac{\partial\psi}{\partial n} = \boldsymbol{n}\cdot\nabla\psi = -\frac{\nabla\psi}{|\nabla\psi|}\nabla\psi = -|\nabla\psi| . \tag{B.2}$$

The magnitude of the gradient is calculated in the variables r and θ:

$$|\nabla\psi(r, \theta)|^2 = \psi_{,r}^2 + \frac{1}{r^2}\psi_{,\theta}^2 \tag{B.3}$$

with

$$\psi_{,r}^2 = \psi_{0,r}^2 + 2\varepsilon\psi_{0,r}\psi_{1,r} + \varepsilon^2[2\psi_{0,r}\psi_{2,r} + \psi_{1,r}^2] + O(\varepsilon^3)$$

and

$$\psi_{,\theta}^2 = \varepsilon^2\psi_{1,\theta}^2 .$$

Expanding the root of (B.3) to second order, we obtain

$$\begin{aligned} |\nabla\psi(r, \theta)| = -\psi_{0,r}(r) - \varepsilon\psi_{1,r}(r, \theta) \\ - \varepsilon^2\left[\psi_{2,r}(r, \theta) + \frac{\psi_{1,\theta}^2(r, \theta)}{2r^2\psi_{0,r}(r)}\right] . \end{aligned} \tag{B.4}$$

It must now be taken into account that for the normalization condition we need the integrand at the location $\psi = \text{const}$. We therefore convert $|\nabla\psi(r, \theta)|$ to $|\nabla\psi(\psi, \theta)|$ and take ψ at the plasma edge. One then has $\hat{r}(\theta) = 1 + \varepsilon^2 \hat{r}_2(\theta)$ and

$$|\nabla\psi(\psi=0,\theta) = -\psi_{0,r}(1) - \varepsilon\psi_{1,r}(1,\theta) - \varepsilon^2[\psi_{2,r}(1,\theta) + \hat{r}_2(\theta)\,\psi_{0,rr}(1)] \quad \text{(B.5)}$$

if use is made of the fact that $\psi_{1,\theta}(1)=0$.

In the line element $dl^2 = dr^2 + r^2\, d\theta^2$ along a curve with ψ=const we express $dr(\psi,\theta)$ as

$$dr(\psi,\theta) = \frac{\partial r}{\partial\psi}\,d\psi + \frac{\partial r}{\partial\theta}\,d\theta = \frac{\partial r}{\partial\theta}\,d\theta\,,$$

from which we arrive at the expression

$$dl = r\left(1 + \frac{1}{r^2}\left(\frac{\partial r}{\partial\theta}\right)^2\right)^{1/2} d\theta\,. \quad \text{(B.6)}$$

As the second term in the radicand is of order ε^4, the line element at the plasma edge up to second order is

$$dl = \hat{r}\,d\theta = (1 + \varepsilon^2 \hat{r}_2(\theta))\,d\theta\,. \quad \text{(B.7)}$$

The last factor of (B.1) in the integrand is finally calculated as

$$\frac{1}{R} = \frac{1}{1+\varepsilon x} = 1 - \varepsilon x + \varepsilon^2 x^2 + O(\varepsilon^3) = 1 - \varepsilon r\cos\theta + \varepsilon^2 r^2 \cos^2\theta + O(\varepsilon^3)\,.$$

At the plasma edge this becomes

$$\frac{1}{R} = 1 - \varepsilon\cos\theta + \varepsilon^2\cos^2\theta\,. \quad \text{(B.8)}$$

Substituting all this in the normalization condition, we obtain in zeroth order

$$1 = -2\pi\,\psi_{0,r}(1) \quad \text{(B.9)}$$

and in first order

$$0 = -\int_0^{2\pi} (\psi_{0,r}(1)\cos\theta - \psi_{1,r}(1,\theta))\,d\theta$$

or, owing to the θ-independence of $\psi_0(r)$,

$$0 = \int_0^{2\pi} \psi_{1,r}(1,\theta)\,d\theta\,. \quad \text{(B.10)}$$

In second order one obtains

$$0 = \int_0^{2\pi} (\psi_{2,r}(1,\theta) + \hat{r}_2(\theta)\,\psi_{0,rr}(1) + \psi_{0,r}(1)\cos^2\theta + \hat{r}_2(\theta)\,\psi_{0,r}(1) - \psi_{1,r}(1)\cos\theta)\,d\theta\,.$$

With

$$\psi_2(r,\theta) = \psi_{20}(r) + \psi_{22}(r)\cos 2\theta$$

and

$$\hat{r}_2(\theta) = -\frac{\psi_2(1,\theta)}{\psi_{0,r}(1)} \quad \text{(B.11)}$$

one obtains

$$0 = 2\pi\,\psi_{20,r}(1) - 2\pi\,\psi_{20}(1)\left(1 + \frac{\psi_{0,rr}(1)}{\psi_{0,r}(1)}\right) + \pi\,\psi_{0,r}(1) - \pi\,\psi_{11,r}(1)\,.$$

Because of (24 a) the term in parentheses is $2\pi\lambda_0$, and so the normalization condition in second order reduces to

$$\psi_{20,r}(1) - 2\pi\lambda_0\,\psi_{20}(1) = \tfrac{1}{2}(\psi_{11,r}(1) - \psi_{0,r}(1))\,. \quad \text{(B.12)}$$

Annex C. Definiteness of the Shafranov Shift

For the following discussion of definiteness it is more convenient to choose the variables λ_0 and ξ. Using (13 c), we obtain

$$x_{s1} = \left[\frac{1}{2} + \frac{4\pi}{\alpha^2}\left(1 - 2\frac{J_1(\alpha)}{\alpha}\right)(\xi - \lambda_0)\right]\beta_P + \frac{1 - \pi\lambda_0}{\alpha^2}\,. \quad \text{(C.1)}$$

Analysis of the β_P-independent term shows that it is positive definite for all $\alpha^2 \in\,]-\infty; j_{11}^2[$.

The β_P-dependent component can simply be estimated. As the scalar pressure cannot take negative values, according to (13 a) it holds that

$$p(r,\theta) = \beta_P\left(\frac{\alpha^2}{2}\,\psi(r,\theta) + \xi\right)\psi(r,\theta) \geq 0\,. \quad \text{(C.2)}$$

As β_P and, in the interior of the plasma, also $\psi(r,\theta)$ are positive it holds that

$$\xi \geq 0 \quad \text{for} \quad \alpha^2 \geq 0 \quad \text{(C.3)}$$

because, if $\xi < 0$, the pressure at the plasma edge, where the flux function vanishes, would take negative values.

Owing to the properties of the Bessel functions the expression $\left(1 - \frac{2}{\alpha}J_1(\alpha)\right)\Big/\alpha^2$ is greater than zero. We thus get for the ξ-dependent term a positive sign and for the ξ-independent component the expression

$$\frac{1}{2} - \frac{2}{\alpha}\,\frac{J_0(\alpha)}{J_1(\alpha)}\left(1 - \frac{2}{\alpha}J_1(\alpha)\right). \quad \text{(C.4)}$$

For $\alpha^2 > 0$ one can then show positive definiteness:

$$x_{s1} \geq 0\,. \quad \text{(C.5)}$$

For negative α^2 the requirement of positive pressure is no longer sufficient for positive Shafranov shift. The expression (C.4) then becomes

$$\frac{1}{2}+\frac{2}{|\alpha|}\frac{I_0(|\alpha|)}{I_1(|\alpha|)}\left(1-\frac{2}{|\alpha|}I_1(|\alpha|)\right). \quad \text{(C.6)}$$

and it is seen that for large $|\alpha|$ the term with the negative sign is dominant. However, the assumption that the pressure rises monotonically in ψ from the edge to the magnetic axis makes the Shafranov shift positive.

Shape of the Contour Lines

In order to see how the flux function behaves in the interior of the plasma, we consider its contour lines $\psi(r, \theta) = c = \text{const.}$

After some rearrangement this relation reads

$$c = \psi_0(x, y) + \frac{\varepsilon}{r}\psi_{11}(x, y)\, x\,. \quad \text{(C.7)}$$

For a given value c we denote the contour line by $R_c(\theta)$ and expand in ε:

$$R_c(\theta) = R_0(\theta) + \varepsilon R_1(\theta) + O(\varepsilon^2)\,. \quad \text{(C.8)}$$

The zeroth order yields an R_0 independent of θ

$$R_0 = \frac{1}{\alpha} J_0^{-1}\left(\frac{c}{c_0} + \frac{\lambda_0}{c_0\alpha^2}\right) \quad \text{(C.9)}$$

with the inverse Bessel function $J_0^{-1}(s)$. The appropriate curve is a circle about the origin with radius R_0 which is described by

$$x^2 + y^2 = R_0^2\,.$$

The first order yields

$$R_1 = \frac{x}{R_0}\frac{\psi_{11}(R_0)}{\psi_{0,r}(R_0)} = k\, x\,. \quad \text{(C.10)}$$

This gives us

$$\begin{aligned} R_c^2(\theta) &= R_0^2 + 2\varepsilon R_0 R_1 + O(\varepsilon^2)\\ &= x^2 + y^2 + 2\varepsilon k R_0 x + O(\varepsilon^2)\\ &= x^2 + y^2 + \varepsilon k' x + O(\varepsilon^2)\end{aligned}$$

with

$$k' = 2\,\frac{\psi_{11}(R_0)}{\psi_{0,r}(R_0)}\,. \quad \text{(C.11)}$$

By shifting the origin of the coordinates by Δx in the direction of the positive x-axis it is possible to make $R_c(\theta)$ constant:

$$\begin{aligned} R_c^2(x, y) &= (x-\Delta x)^2 + y^2 + \varepsilon k'(x-\Delta x) + O(\varepsilon^2)\\ &= x^2 + y^2 + (\varepsilon k' - 2\Delta x)\, x - \varepsilon k'\Delta x + O(\varepsilon^2)\,.\end{aligned}$$

We choose Δx such that the term linear in x vanishes.

$$\Delta x = \tfrac{1}{2}\varepsilon k'\,. \quad \text{(C.12)}$$

Consequently, all terms linear in ε vanish, leaving

$$R_c^2 = x^2 + y^2 + O(\varepsilon^2) = R_0^2 + O(\varepsilon^2)\,.$$

This means that the set of curves R_c describes circles with radius R_0 that are shifted by Δx.

Annex D

It is assumed that a_{33} is non-zero, which will be verified later. In principle, we can then calculate b_1 and assume it to be known in the following. We are now left with the reduced system of equations

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\lambda_2\\ a_2\end{pmatrix} = \begin{pmatrix} g_1\\ g_2\end{pmatrix}. \quad \text{(D.1)}$$

To determine the coefficients of the matrix, we split off from the inhomogeneity of the integral equation the terms not containing λ_2 and a_2. This leaves integral equations whose inhomogeneous terms are linear in λ_2 and a_2 and whose solutions are denoted by $\psi_{20}^*(r)$ and $\psi_{22}^*(r)$. Terms without λ_2 and a_2 are abbreviated by:

$$\psi_{20}(r) = \psi_{20}^*(r) + \dots\,, \quad \text{(D.2a)}$$

$$\psi_{22}(r) = \psi_{22}^*(r) + \dots\,. \quad \text{(D.2b)}$$

It then follows that

$$\begin{aligned} \psi_{20}^*(r) = \alpha^2 \int_0^1 \left(\ln\frac{8}{\varepsilon} - 2 + K_0(r, r')\right)\psi_{20}^*(r')\, r'\,\mathrm{d}r'\\ + 2\pi\lambda_0\left(\ln\frac{8}{\varepsilon} - 2\right)\psi_{20}^*(1) \quad \text{(D.3a)}\\ + \lambda_2\int_0^1\left(\ln\frac{8}{\varepsilon} - 2 + K_0(r, r')\right) r'\,\mathrm{d}r' + a_2,\end{aligned}$$

and

$$\psi_{22}^*(r) = \alpha^2\int_0^1 K_2(r, r')\,\psi_{22}^*(r')\, r'\,\mathrm{d}r' + \frac{\pi}{2}\lambda_0\,\psi_{22}^*(1)\, r^2\,. \quad \text{(D.3b)}$$

The solution for $\alpha \neq 0$ reads

$$\psi_{22}^*(r) = c_{20}^* J_0(\alpha r) - \frac{\lambda_2}{\alpha^2} \tag{D.4a}$$

with the constant c_{20}^*

$$c_{20}^* = \frac{1}{N}\left(\frac{M}{\alpha^2}\lambda_2 + \alpha_2\right) \tag{D.4b}$$

and with

$$M = 1 - 2\pi\lambda_0\left(\ln\frac{8}{\varepsilon} - 2\right)$$

and

$$N = M J_0(\alpha) - \left(\ln\frac{8}{\varepsilon} - 2\right)\alpha J_1(\alpha)\,.$$

The solution of the problem for $\psi_{22}^*(r)$ is

$$\psi_{22}^*(r) \equiv 0\,. \tag{D.5}$$

With these results we write the normalization and subsidiary conditions

$$\text{(N)}\quad -c_{20}^*\,\alpha J_1(\alpha) - 2\pi\lambda_0\left[c_{20}^* J_0(\alpha) - \frac{\lambda_2}{\alpha^2}\right] = \ldots \tag{D.6a}$$

$$\text{(B)}\quad c_{20}^* J_0(\alpha) - \frac{\lambda_2}{\alpha^2} = \ldots\,. \tag{D.6b}$$

In matrix notation we then have

$$\begin{pmatrix} -\dfrac{MQ}{\alpha^2 N} + \dfrac{2\pi\lambda_0}{\alpha^2}, & -\dfrac{Q}{N} \\[2ex] \dfrac{M J_0(\alpha)}{\alpha^2 N} - \dfrac{1}{\alpha^2}, & \dfrac{J_0(\alpha)}{N} \end{pmatrix}\begin{pmatrix}\lambda_2 \\ a_2\end{pmatrix} = \ldots \tag{D.7}$$

with

$$Q = \alpha J_1(\alpha) + 2\pi\lambda_0 J_0(\alpha)\,.$$

Neither the factor N nor the right-hand side has poles. None of the coefficients a_{ij} $(i, j = 1, 2)$ vanishes from the matrix equation (D.1). The determinant of the matrix A

$$\det A = -\frac{J_1(\alpha)}{\alpha N}$$

is regular ($\neq 0$), which ensures that the system of inhomogeneous equations always has a unique solution.

The limit $\alpha \to 0$ yields

$$\lim_{\alpha\to 0}(\det A) = -\frac{1}{2N} = \frac{1}{4\ln\dfrac{8}{\varepsilon} - 6}\,.$$

This demonstrates that λ_2, a_2, and b_1 can always be chosen such that they satisfy all conditions that we have imposed on $\psi_2(r, \theta)$.

Solution for $\psi_{22}(r)$

We solve the integral equation (59c). It is of the form

$$\psi_{22}(r) = \alpha^2 \int_0^1 K_2(r, r')\,\psi_{22}(r')\,r'\,dr' + \frac{\pi}{2}\lambda_0\psi_{22}(1)\,r^2 + A(r).$$

The inhomogeneity $A(r)$ is a polynomial in β_P:

$$A(r) = A_3\beta_P^3 + \beta_P^2 A_2(r) + \beta_P A_1(r) + A_0(r)\,. \tag{D.8}$$

As the integral equation for $\psi_{22}(r)$ is linear in $\psi_{22}(r)$, this division can be adopted for the solution:

$$\psi_{22}(r) = \beta_P^3\psi_{223}(r) + \beta_P^2\psi_{222}(r) + \beta_P\psi_{221}(r) + \psi_{220}(r). \tag{D.9}$$

The individual components of the solution then satisfy the integral equations

$$\psi_{22i}(r) = \alpha^2\int_0^1 K_2(r, r')\,\psi_{22i}(r')\,r'\,dr' + \frac{\pi}{2}\lambda_0\psi_{22i}(1)\,r^2 + A_i(r), \quad i = 0, 1, 2, 3 \tag{D.10}$$

with the inhomogeneities

$$A_3(r) = \int K_2(r, r')\,[4\pi c_0 \nu\alpha J_1(\alpha r') - 2\nu r']\,r'^2\,dr' \tag{D.11a}$$

$$A_2(r) = \int K_2(r, r')\,[\tfrac{1}{2}c_0\alpha^3(1 - r'^2)J_1(\alpha r') - 5\pi c_0\nu\alpha J_1(\alpha r') + \tfrac{5}{2}\nu r']\,r'^2\,dr' + \pi c_0\nu\alpha r\int K_1(r, r')\,J_1(\alpha r')\,r'\,dr' - \frac{\nu(1-\pi\lambda_0)}{2\alpha^2}r^2, \tag{D.11b}$$

$$A_1(r) = \int K_2(r, r')\left[\pi c_0\alpha(\nu - \lambda_0)\,J_1(\alpha r') + \tfrac{1}{8}c_0\alpha^3(r'^2 - 1)\,J_1(\alpha r') + \tfrac{1}{2}c_0\alpha^2 r' J_0(\alpha r') - \frac{\nu}{2}r'\right]r'^2\,dr'$$
$$+ r\int K_1(r, r')\,[\tfrac{1}{2}c_0\alpha^2 r' J_0(\alpha r') + \tfrac{1}{8}c_0\alpha^3(1 - r'^2)\,J_1(\alpha r') - \pi c_0\nu\alpha J_1(\alpha r')]\,r'\,dr' + \left(-\frac{1}{16\pi} + \frac{\nu(1-\pi\lambda_0)}{2\alpha^2}\right)r^2, \tag{D.11c}$$

and

$$A_0(r) = \int K_2(r,r')\left[\tfrac{1}{16}\, c_0 \alpha^2 r' J_0(\alpha r') + \tfrac{1}{4}\pi c_0 \lambda_0 \alpha J_1(\alpha r')\right] r'^2\, dr' - \tfrac{1}{4}\pi c_0 \lambda_0 \alpha r \int K_1(r,r') J_1(\alpha r') r'\, dr'$$
$$- \frac{1}{16}\left(c_0 J_0(\alpha r) - \frac{\lambda_0}{\alpha^2}\right) r^2 - \frac{1}{32\pi}\left(3 \ln \frac{8}{\varepsilon} - 4\right) r^2 + \frac{\lambda_0 (1-\pi\lambda_0)}{8\alpha^2} r^2 . \qquad \text{(D.11d)}$$

We solve the integral equation by using the known differential operator L_2 to transform them to the equivalent differential equations

$$[r(\psi_{22i}(r)_{,r}]_{,r} + \left(\alpha^2 r - \frac{4}{r}\right)\psi_{22i} = L_2[A_i(r)] , \qquad \text{(D.12a)}$$

taking their respective boundary conditions into account:

$\psi_{22i}(0)$ finite

and

$$(r^2 \psi_{22i})_{,r}(1) = 2\pi \lambda_0 \psi_{22i}(1) + (r^2 A_i)_{,r}(1) . \qquad \text{(D.12b)}$$

The right-hand sides of the differential equations read

$$L_2[A_3(r)] = -4\pi c_0 \nu \alpha r^2 J_1(\alpha r) + 2\nu r^3 , \qquad \text{(D.13a)}$$

$$L_2[A_2(r)] = 2\pi c_0 \nu r J_0(\alpha r) \qquad \text{(D.13b)}$$
$$+ c_0\left[\frac{1}{2}\alpha^3 r^4 + \left(4\pi\nu\alpha - \frac{\alpha^3}{2}\right) r^2 - \frac{4\pi\nu}{\alpha}\right] J_1(\alpha r) - \tfrac{5}{2}\nu r^3,$$

$$L_2[A_1(r)] = c_0\left[-\frac{5}{4}\alpha^2 r^3 + \left(\frac{\alpha^2}{4} - 2\pi\nu\right) r\right] J_0(\alpha r)$$
$$+ c_0\left[\pi\lambda_0 \alpha r^2 - \frac{\alpha}{2} + \frac{4\pi\nu}{\alpha}\right] J_1(\alpha r) + \frac{1}{2}\nu r^3, \qquad \text{(D.13c)}$$

$$L_2[A_0(r)] = -\frac{\pi c_0 \lambda_0}{2} r J_0(\alpha r) + \frac{\pi c_0 \lambda_0}{\alpha} J_1(\alpha r)$$
$$+ \tfrac{1}{4} c_0 \alpha r^2 J_1(\alpha r) , \qquad \text{(D.13d)}$$

and the general inhomogeneous solutions are

$$\psi_{223}(r) = (\pi c_0 \nu r^2 - c_{223}) J_0(\alpha r) + \frac{2 c_{223}}{\alpha r} J_1(\alpha r) + \frac{2\nu}{\alpha^2} r^2 . \qquad \text{(D.14a)}$$

$$\psi_{222}(r) = \left[c_0\left(-\frac{\alpha^2}{16} r^4 + \left(\frac{\alpha^2}{8} - \pi\nu\right) r^2\right) - c_{222}\right] J_0(\alpha r)$$
$$+ \left\{c_0\left[\frac{\alpha}{8} r^3 + \frac{\pi\nu}{\alpha} r\right] + \frac{2 c_{222}}{\alpha r}\right\} J_1(\alpha r) - \frac{5\nu}{2\alpha^2} r^2, \qquad \text{(D.14b)}$$

$$\psi_{221}(r) = [-\tfrac{1}{4}\pi c_0 \lambda_0 r^2 - c_{221}] J_0(\alpha r) + \frac{\nu}{2\alpha^2} r^2$$
$$+ \left\{c_0\left[-\frac{5}{24}\alpha r^3 + \left(\frac{\alpha}{8} - \frac{\pi\nu}{\alpha}\right) r\right] + \frac{2 c_{221}}{\alpha r}\right\} J_1(\alpha r) \qquad \text{(D.14c)}$$

$$\psi_{220}(r) = [-\tfrac{1}{16} c_0 r^2 - c_{220}] J_0(\alpha r) + \left[-\frac{\pi c_0 \lambda_0}{4\alpha} r + c_{220} \frac{2}{\alpha r}\right] J_1(\alpha r) . \qquad \text{(D.14d)}$$

The constants c_{22i} are determined from the boundary conditions:

$$c_{223} = \frac{1}{\alpha J_1(\alpha) - 2\pi\lambda_0 J_2(\alpha)}\, \nu \left\{\frac{1}{\alpha^2}(2\pi^2 \lambda_0^2 - 8) + \frac{1}{2}\right\}, \qquad \text{(D.15a)}$$

$$c_{222} = \frac{1}{\alpha J_1(\alpha) - 2\pi\lambda_0 J_2(\alpha)} \left\{-\frac{2\pi^2 \lambda_0^2 \nu}{\alpha^2} - \frac{1}{8}\lambda_0 (1-\pi\lambda_0) + \frac{\alpha^2}{32\pi} - \frac{\nu}{2} - \frac{1}{4\pi} + 8\frac{\nu}{\alpha^2}\right\}, \qquad \text{(D.15b)}$$

$$c_{221} = \frac{1}{\alpha J_1(\alpha) - 2\pi\lambda_0 J_2(\alpha)} \left\{-\frac{\pi^2 \lambda_0^2}{2\alpha^2} + \frac{\pi \lambda_0^2}{\alpha^2} - \frac{\lambda_0}{8} + \frac{1}{6\pi}\right\}, \qquad \text{(D.15c)}$$

$$c_{220} = \frac{1}{\alpha J_1(\alpha) - 2\pi\lambda_0 J_2(\alpha)} \left\{\frac{3}{8}\frac{\lambda_0}{\alpha^2}(2-\pi\lambda_0) - \frac{3\ln\frac{8}{\varepsilon} - 4}{8\pi}\right\}. \qquad \text{(D.15d)}$$

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